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Variational principles and gauge theories of gravitation

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Abstract. Attention is drawn to difficulties associated with gravitational theories based on Lagrangians formed from quadratic invariants of the Riemann tensor, and in particular with the Palatini variational method used by Yang in a gauge theory of gravitation. It is pointed out that the Yang theory is not mathematically well founded.

1. Introduction

In a recent paper Fairchild (1976) has attempted to develop a gauge theory of gravitation based on the recent work of Yang (1974) which, in turn, employs a variational principle discussed by the present author some years ago (see Stephenson 1958). The purpose of this paper is to draw attention to the difficulties associated with the particular type of variation used, and to point to other difficulties inherent in the Stephenson–Yang theory (as it is referred to by Fairchild).

There are two basic ways of obtaining field equations from a variational principle, referred to by Buchdahl (1960) as the g-variation and P-variation methods. We now examine each of these in turn.

2. g-variations

In this approach we consider a Riemann space in which the connection is the Christoffel connection formed from the metric tensor $g_{i\kappa}$. The Lagrangian L, which is constructed the principle $\delta \int L\sqrt{-g}\,\mathrm{d}\tau = 0$, where the variation is with respect to the $g_{i\kappa}$. In the case when $L = R$, where R is the scalar curvature, we obtain the field equations $R_{i\kappa} - \frac{1}{2}g_{i\kappa}R = 0$ of general relativity. g-variations of Lagrangians formed from quadratic invariants of the Riemann tensor and its contractions have been studied extensively by many authors for various reasons since the early days of relativity. It is not appropriate to give a survey of the subject here, but the work is associated with the names of Eddington, Weyl, Pauli, Lanczos, Gregory, Buchdahl and many others (for detailed references and further work, see Stephenson 1958, 1969, Lovelock 1970, and Bicknell 1974). Much of this effort has been directed towards developing alternative field equations for the gravitational field, and also towards trying to obtain a unification of the gravitational and electromagnetic fields. In recent years, the question of the quantization of the gravitational field has attracted interest in quadratic Lagrangians and the work of DeWitt (1964, 1967, 1975) is especially relevant. It is well known that as far as the quantization of the weak field approximation of the general relativity field equations is concerned the theory is not renormalizable by the ordinary techniques of quantum field theory. The inclusion of quadratic invariants of the Riemann tensor in the Lagrangian for the gravitational field may improve the situation, but the physical interpretation of such Lagrangians is not clear. (For a survey of this problem the reader should also consult Salam (1971), Deser and Vanieuwenhuyzen (1974), and Isham *et al* (1975).)

Leaving aside for the moment the motivation for discussing quadratic Lagrangians, we now come to two particular mathematical results which are relevant here. We define

$$I_1=\int R^2\sqrt{-g}\,\mathrm{d}\tau, \qquad I_2=\int R_{i\kappa}R^{i\kappa}\sqrt{-g}\,\mathrm{d}\tau, \qquad I_3=\int R^{i}{}_{j\kappa l}R_i{}^{j\kappa l}\sqrt{-g}\,\mathrm{d}\tau. \tag{1}$$

Then Lanczos (1938) has shown that only two of these three integrals are independent under g-variations in a four-dimensional space and that

$$\delta(I_1-4I_2+I_3)=0. \tag{2}$$

Furthermore, it is important to realize that the field equations obtained from these three variational principles are all fourth-order partial differential equations for the $g_{i\kappa}$, unlike the field equations of general relativity which are second order. Using Lanczos's result, the field equations obtained from a linear combination of the three quadratic Lagrangians (1) may therefore be derived from a Lagrangian which is just a linear combination of R^2 and $R_{i\kappa}R^{i\kappa}$, and Bicknell (1974) has discussed fully the implications of taking such a Lagrangian as a basis for a theory of gravitation alone. His conclusions are that the field equations based on R^2 possess physically acceptable solutions, but that when a matter or source field is introduced into the Lagrangian the results differ widely from those predicted by general relativity. Purely quadratic Lagrangians, he concludes, are not suitable for constructing viable theories of gravitation. The fact that in the absence of a source field in the Lagrangian the field equations obtained from $\delta\int R^2\sqrt{-g}\,\mathrm{d}\tau=0$ are physically acceptable is part of a general result obtained earlier (see Stephenson 1969). For suppose we consider some function of the scalar curvature, $F(R)$, and consider the g-variation of

$$\delta\int F(R)\sqrt{-g}\,\mathrm{d}\tau=0. \tag{3}$$

We can now ask what conditions the function F has to satisfy in order that the field equations so obtained are satisfied by the empty space field equations $R_{i\kappa}=0$ of general relativity. At least in this way we shall guarantee that the generalized field equations will contain the physically significant solutions of conventional gravitational theory. The result obtained is that $F(R)$ should be a function of class C^3 at least and that $F(0)=0$. For example, $\delta\int(\sin R)\sqrt{-g}\,\mathrm{d}\tau=0$ will lead to field equations which are satisfied by $R_{i\kappa}=0$ since $\sin R$ is C^∞ and $\sin 0=0$. However, $\delta\int \mathrm{e}^R\sqrt{-g}\,\mathrm{d}\tau=0$ does not satisfy the conditions. It is now clear why $\delta\int R^2\sqrt{-g}\,\mathrm{d}\tau=0$, and indeed $\delta\int R^n\sqrt{-g}\,\mathrm{d}\tau=0$ ($n\geqslant 2$) yield field equations which are satisfied by $R_{i\kappa}=0$. However, the objections to quadratic Lagrangians combined with a matter Lagrangian as raised by Bicknell (1974) still remain. To overcome these objections it would seem necessary to include the linear term R in the Lagrangian. As far as is known, there appears to be no basic objection to a theory based on the g-variation of

$$\delta \int (\alpha R + \beta R^2 + \gamma R_{i\kappa} R^{i\kappa})\sqrt{-g}\, d\tau = 0, \tag{4}$$

where α, β and γ are suitably chosen constants. The difficulty here may be that there exist a number of non-physical type solutions and this possibility has been discussed by Buchdahl (1962), Thompson (1975a,b), Pavelle (1975) and Fairchild (1976) in relation to various theories.

3. *P*-variations

We now come to the method of variation due to Palatini (1919). In this case the space at the outset is not assumed to be Riemannian and the metric tensor $g_{i\kappa}$ and the affine connection $\Gamma^s_{i\kappa}$ (assumed symmetric) are varied independently. P-variations of the three quadratic Lagrangians R^2, $R_{i\kappa}R^{i\kappa}$, $R^i{}_{j\kappa l}R_i{}^{j\kappa l}$ have been discussed (see Stephenson 1958, 1959, 1960 and Buchdahl 1960). The field equations fall into two sets—one set from the variation with respect to the $g_{i\kappa}$ and the other from the variation with respect to the $\Gamma^s_{i\kappa}$. Now Yang (1974) in attempting to set up a gauge theory of gravitation took just one set of the field equations obtained from the P-variation of $\delta \int R^i{}_{j\kappa l}R_i{}^{j\kappa l}\sqrt{-g}\, d\tau = 0$ and ignored the other set (namely that obtained from the variation with respect to the $\Gamma^s_{i\kappa}$). Apart from this lack of consistency, observed also by Fairchild (1976), the P-variation method has been severely and properly criticized by Buchdahl (1960) (see also Stephenson 1959). It is only for the linear Lagrangian R that the g-variation and the P-variation methods lead to the same field equations. In all other cases the results are different, and as Buchdahl has shown lead to most strange results—so much so that the technique must be regarded as unsuitable as a basis for a physical theory. One example given by Buchdahl will suffice to illustrate the type of difficulty which can arise: if $L = R_{i\kappa}R^{i\kappa}$ then the field equations in a V_4 are found to be

$$R_{is}R^{s\kappa} + R_{si}R^{\kappa s} - \tfrac{1}{2}\delta^\kappa_i R_{sp}R^{sp} = 0 \qquad (R^{\underline{i\kappa}}\sqrt{-g})_{;s} = 0, \tag{5}$$

where the semicolon denotes covariant differentiation with respect to the symmetric connection $\Gamma^s_{i\kappa}$. These two sets of equations are satisfied by any set of 40 functions $\Gamma^s_{i\kappa}$ which make $R_{i\kappa}$ vanish. The $g_{i\kappa}$ remain arbitrary and undetermined. Another objection to the P-variation method as applied to quadratic Lagrangians is founded on the Weyl gauge group in which $g_{i\kappa} \to \phi(x^s)g_{i\kappa}$, $\Gamma^s_{i\kappa} \to \Gamma^s_{i\kappa}$, where $\phi(x^s)$ is an arbitrary function of the coordinates. The quadratic Lagrangian densities $R^2\sqrt{-g}$, $R_{i\kappa}R^{i\kappa}\sqrt{-g}$, $R^i{}_{j\kappa l}R_i{}^{j\kappa l}\sqrt{-g}$ are all invariant under these gauge transformations (see Stephenson 1960) and as a consequence of this extra invariance the field equations derived from the $g_{i\kappa}$ part of the variation have vanishing trace (see, for example, the first set of equations in (5)). This presents a problem, even allowing for the strange properties noted by Buchdahl, since if one wishes to include a matter Lagrangian in the usual way then it too would have to have zero trace. Such a restriction is physically unacceptable.

4. Conclusion

The aim of this short paper has been to point out certain difficulties associated with theories based on variational principles. It has not been possible, nor considered desirable, to give a complete survey of the field, and consequently many references to the work of other authors have necessarily been omitted.

To summarize, it does seem that in order to obtain a satisfactory set of field equations via the g-variation method the linear term R must be included in the Lagrangian, and that no clear prescription for including quadratic terms is available either from classical or quantum arguments. However, we see that the P-variation method applied to quadratic Lagrangians is highly dubious, and further that the Yang theory is mathematically inconsistent, based as it is on only one set of the relevant field equations obtained from a P-variational principle. P-variations should be used with extreme caution on any Lagrangian other than R.

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